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# On Approximation Operators of the Bernstein Type

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## 1. INTRODUCTION

Let the sequence $\{\lambda_i\}(i \geq 0)$ satisfy

$$0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots \infty, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty, \tag{1.1}$$

and define (the divided difference)

$$[x^{\lambda_m}, \ldots, x^{\lambda_m}] = \sum_{i=m}^{\infty} x^{\lambda_i}/\omega'_{nm}(\lambda_i), \qquad 0 \leq m \leq n = 0, 1, 2, \ldots,$$

where $\omega_{nm}(x) = (x - \lambda_m)\ldots(x - \lambda_n)$, $0 \leq m \leq n = 0, 1, 2, \ldots$. Denote

$$p_{nm}(x) = (-1)^{n-1} \lambda_m \ldots \lambda_{n-1} [x^{\lambda_m}, \ldots, x^{\lambda_n}], \qquad 0 \leq m < n, p_{nn}(x) = x^{\lambda_n}$$

and

$$\alpha_{nm} = \{(1 - \lambda_1/\lambda_m)\ldots(1 - \lambda_1/\lambda_{n-1})\}^{1/\lambda_1}, \qquad 0 \leq m < n, \alpha_{nn} = 1.$$

It is well known that $p_{nm}(x) \geq 0$ for $0 \leq m \leq n = 0, 1, 2, \ldots$ and $0 \leq x \leq 1$.

With a function $f(t)$, bounded in $[0, 1]$, we associate the operators

$$L_m(f, x) = \sum_{n=m}^{\infty} p_{nm}(x) f(\alpha_{nm}), \qquad m \geq 0.$$

These operators, which generalize the Bernstein power-series of Meyer–König and Zeller [*7*], were first introduced by Jakimosvki and the author in [*4*], where some approximation properties were stated without proof. (For proofs see [*5*]). More recently, these operators were redefined and their approximation properties, were studied independently and from a different point of view by Feller [*2*].

It is our purpose here to discuss the approximation properties of the derivatives of $L_m(f, x)$.

## 2. AUXILIARY LEMMAS

We make use of the following lemmas.

LEMMA A. *Let $f(t)$ be bounded in* $[0,1]$. *Then* (i) *At each point of continuity* $t=x$, $0<x\leqslant 1$, of $f(t)$

$$\lim_{m\to\infty} L_m(f,x)=f(x). \tag{2.1}$$

(ii) *If $f(t)$ is continuous in $[a,b]$, $0<a<b\leqslant 1$, then* (2.1) *holds uniformly in* $a\leqslant x\leqslant b$. (iii) *If $f(t)$ is continuous in* $[0,b]$, $0<b\leqslant 1$, *then* (2.1) *holds uniformly in* $0<x\leqslant b$. (iv) *For* $0<x\leqslant 1$, *we have* $L_m(1,x)\equiv 1$, $m\geqslant 0$.

Lemma A is Theorem 4.1 of [*4*] and is proved in [*5*].

LEMMA B. *For* $0<x\leqslant 1$ *and* $0\leqslant m\leqslant n=0,1,2,\ldots$, *we have*

$$\frac{d}{dx}p_{nm}(x)=x^{-1}[\lambda_n p_{nm}(x)-\lambda_{n-1}p_{n-1,m}(x)] \tag{2.2}$$

*and*

$$\frac{d}{dx}p_{nm}(x)=x^{-1}\lambda_m[p_{nm}(x)-p_{n,m+1}(x)]. \tag{2.3}$$

($p_{nk}(x)=0$ *for* $n<k$).

Lemma B was used by the author several times in the past (see for example [*4*], (4.5)).

LEMMA C. *Let $f(t)$ be bounded in* $[0,1]$. *Then for* $0<x\leqslant 1$ *and every* $m\geqslant 0$, *we have*

$$\frac{d}{dx}L_m(f,x)=\sum_{n=m}^{\infty}\frac{d}{dx}p_{nm}(x)f(\alpha_{nm}). \tag{2.4}$$

*Proof.* Let $M=\sup_{0\leqslant t\leqslant 1}|f(t)|$. By (2.3), we have for every $0<x\leqslant 1$,

$$\left|\frac{d}{dx}p_{nm}(x)f(\alpha_{nm})\right|\leqslant M\lambda_m x^{-1}[p_{nm}(x)+p_{n,m+1}(x)].$$

By Lemma A, the series $\sum_{n=m}^{\infty}p_{nm}(x)$ converges to the continuous function 1 for $0<x\leqslant 1$. Since $p_{nm}(x)\geqslant 0$ for $0\leqslant m\leqslant n=0,1,2,\ldots$ and $0<x\leqslant 1$, the convergence is monotonic and thus uniform in $0<\delta\leqslant x\leqslant 1$ for any fixed $\delta>0$. Therefore $L_m(f,x)$ is differentiable and (2.4) holds.

## 3. MAIN RESULTS

THEOREM 1. *Let $f(t)$ be a continuously differentiable function in* $0\leqslant t\leqslant 1$. *Then for every fixed* $\delta>0$,

$$\lim_{m\to\infty}\frac{d}{dx}L_m(f,x)=f'(x) \text{ uniformly in } \delta\leqslant x\leqslant 1. \tag{3.1}$$

*Proof.* Assume first that $\lambda_1 = 1$. By (2.4) and (2.2). we have

$$\frac{d}{dx} L_m(f, x) = x^{-1} \sum_{n=m}^{\infty} [\lambda_n p_{nm}(x) - \lambda_{n-1} p_{n-1,m}(x)] f(\alpha_{nm}). \tag{3.2}$$

Now,

$$\sum_{n=m}^{N} [\lambda_n p_{nm}(x) - \lambda_{n-1} p_{n-1,m}(x)] f(\alpha_{nm})$$

$$= \sum_{n=m}^{N} p_{nm}(x) \lambda_n [f(\alpha_{nm}) - f(\alpha_{n+1,m})] + \lambda_N p_{Nm}(x) f(\alpha_{N+1,m}),$$

and since, for every $m \geqslant 1$ and $0 < x \leqslant 1$ we have $\lambda_N p_{Nm}(x) \to 0$ as $N \to \infty$ (see [*3*] Satz 1), and $f(t)$ is bounded, we obtain

$$\sum_{n=m}^{\infty} [\lambda_n p_{nm}(x) - \lambda_{n-1} p_{n-1,m}(x)] f(\alpha_{nm}) = \sum_{n=m}^{\infty} p_{nm}(x) \lambda_n [f(\alpha_{nm}) - f(\alpha_{N+1,m})].$$

Thus, by (3.2),

$$\frac{d}{dx} L_m(f, x) = x^{-1} \sum_{n=m}^{\infty} p_{nm}(x) \lambda_n [f(\alpha_{nm}) - f(\alpha_{n+1,m})]$$

$$= x^{-1} \sum_{n=m}^{\infty} p_{nm}(x) \lambda_n (\alpha_{nm} - \alpha_{n+1,m}) f'(\theta_{nm}) \tag{3.3}$$

$$= x^{-1} \sum_{n=m}^{\infty} p_{nm}(x) \alpha_{nm} f'(\theta_{nm}),$$

where $\alpha_{n+1,m} < \theta_{nm} < \alpha_{nm}$, $n \geqslant m \geqslant 0$.

Since $f'(t)$ is continuous in $[0,1]$, for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|f'(t_1) - f'(t_2)| < \epsilon$ provided $|t_1 - t_2| < \delta$. Take $m_0$ sufficiently large so that $\lambda_{m_0}^{-1} < \delta$; then for $n \geqslant m \geqslant m_0$ we have

$$0 \leqslant \alpha_{nm} - \theta_{nm} \leqslant \alpha_{nm} - \alpha_{n+1,m} = \lambda_n^{-1} \alpha_{nm} \leqslant \lambda_n^{-1} < \delta.$$

Consequently for $m \geqslant m_0$,

$$\left| \frac{d}{dx} L_m(f, x) - x^{-1} L_m(t f'(t), x) \right| \leqslant x^{-1} \sum_{n=m}^{\infty} p_{nm}(x) \alpha_{nm} |f'(\theta_{nm}) - f'(\alpha_{nm})| \tag{3.4}$$

$$\leqslant \epsilon x^{-1} \sum_{n=m}^{\infty} p_{nm}(x) = \epsilon x^{-1}.$$

It follows by Lemma A (ii) that for every fixed $\delta > 0$,

$$\lim_{m \to \infty} L_m(t f'(t), x) = x f'(x), \quad \text{uniformly in } \delta \leqslant x \leqslant 1;$$

hence by (3.4)

$$\lim_{m\to\infty} \frac{d}{dx} L_m(f, x) = f'(x), \quad \text{uniformly in } \delta \leqslant x \leqslant 1.$$

This concludes our proof for the case $\lambda_1 = 1$. If $\lambda_1 \neq 1$, apply Theorem 1 (which is already proved for $\lambda_1 = 1$) to the sequence $\{\lambda_i \lambda_1^{-1}\}(i \geqslant 0)$ and the function $g(t) = f(t^{1/\lambda_1})$.

*Remark.* For another kind of Bernstein type approximation operators known as the generalized Bernstein polynomials (see [*6*]), a theorem similar to Theorem 1 was given by Badaljan [*1*].

THEOREM 2. *Let $f(t)$ be bounded in* [0, 1]. *If $f(t)$ is nondecreasing* (*non-increasing*) *in* [0, 1], *then* (*for each fixed $m \geqslant 0$*) *so is $L_m(f, x)$*.

*Proof.* Since the first equality in (3.3) is proved for every function bounded in [0, 1], our result follows immediately by (3.3).

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